International Journal of Theoretical Physics, Vol. 18, No. 3, 1979

Geometric Quantization and Internal Symmetry

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Received July 19, 1978

As a part of an attempt to geometrize physics, internal symmetries in the covariant classification of matter by its $T_{\mu\nu}$ type are considered in relation to phase transformations generated by complex and quaternionic structures on space-time. The Rainich theory of electromagnetism and neutrinos is compared with the theory of $U(1) \times SO(1, 3)$ torsional gauge fields, and extended to the quaternionic case. It is shown by the Kostant technique of geometric quantization that complex and quaternionic phase transformations for an Einstein space are associated with one-dimensional and three-dimensional harmonic oscillators.

1. INTRODUCTION

This paper extends an attempt to see how much of quantum physics can be geometrized (Lloyd-Evans, 1976a, b, 1977, 1978) to consideration of the role of $T_{\mu\nu}$ and its symmetries (Plebanski, 1964) in geometric quantization. In considering the physical significance of these symmetries Plebanski's use of generalized duality rotations of $U_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{4}T_\alpha{}^\alpha g_{\mu\nu}$ to define formal electromagnetic fields is regarded as an extension of the Rainich unification of gravity and electromagnetism, and Rainich theory is compared in Section 2 with the gauge theory of electromagnetic and weak interactions, emphasizing the nonuniqueness of the former and tentatively suggesting modification to the latter.

The $U(3)$ generalization of Rainich theory is then compared in Section 3 with the theory of quaternionic phase transformations, and in Section 4 Kostant's approach to quantization (Kostant, 1970a, b; Simms and Woodhouse, 1976) applied to this situation gives an outline resemblance to string theory.

An essential feature of this geometric approach is a major departure from current practice in physics by classifying matter in terms of the $T_{\mu\nu}$

type and representations of its symmetry group and of the homogeneous Lorentz group. Thus, rather than calculate $T_{\mu\nu}$ by the methods of Lagrangian quantum field theory for Poincaré covariant matter types, physical $T_{\mu\nu}$ together with its geometric interpretation via the Einstein tensor $G_{\mu\nu}$ is the starting point of the classification scheme. Use of the $T_{\mu\nu}$ classification appears to require an unconventional method of quantization, this work being part of such an approach, with added motivation coming from the absence of a global Fourier decomposition of fields over curved space-time and the lack of a local definition of the energy-momentum vector P_μ.

The basic requirements of the geometric quantization scheme considered previously by the author (1976b) are the existence of a complex structure J and of a symmetric second-rank tensor on space-time that is Hermitian with respect to J. The complex structure used corresponds to that which appears in the definition of two-component spinors, and defines a reduction $GL(4, \mathbb{R})/GL(2, \mathbb{C})$ of the bundle of general linear frames to that of complex linear frames, whilst the complex scale factor defines a further reduction to $SL(2, \mathbb{C})$ null frames (and spin frames up to a sign). After a Wick rotation, this complex structure, together with a phase rotation generated by itself, determines the conformal structure of space-time. In particular, the metric tensor $g_{\mu\nu}$ is Hermitian with respect to this choice of complex structure, and this leads to a finite-dimensional application of Segal's method (Segal, 1960) to define creation and annihilation operators for vectors and spinors.

Ordinary relativistic quantum mechanics is described geometrically by systems of imprimitivity for the structure group $\mathbb{R}^4 \circledS SL(2, \mathbb{C})$ of the bundle of affine spin frames, and this was shown (Lloyd-Evans, 1979) to lead to the coupling of matter to geometry by means of preferred affine frames, and to the existence of a gravitational field obeying the Einstein–Cartan field equations. In addition to the $GL(4, \mathbb{R})$ group of linear frame transformations, a second $GL(4, \mathbb{R})$ group appears at this stage, namely that of infinitesimal coordinate transformations, which is associated geometrically with the inhomogeneous part of the general affine connection. The momentum vector P_μ of a free particle determines a preferred coordinate frame, but once the existence of gravity has been shown, the Poincaré invariant description of matter ceases to be sufficiently general. If the Plebanski classification of $T_{\mu\nu}$ is adopted instead, then it is apparent that free quantum mechanical particles correspond only to the highly specialized types $[T\text{-}3S]_{(2)}$ for massive particles, and $[4N]_{(2)}$ for massless.

The generalization of the energy-momentum vector P_μ of a massive particle in flat space to curved space is an eigenvector of $T_{\mu\nu}$ pointing in the same direction; and in Section 2 of this paper the Plebanski classification of $T_{\mu\nu}$ is used to show how this tensor determines a vierbein field, and how this is related to the complex structure J. This leads to the possibility of three linearly independent complex structures in the case $[T\text{-}3S]_{(2)}$, and to a comparison of J-phase transformations with duality transformations, making particular reference to the relationship between the $U(1)$ gauge theory of electromagnetism and Rainich theory. This emphasizes the distinction between contributions to electromagnetism of types $[2T\text{-}2S]_{(2)}$ and $[4N]_{(2)}$, of which the latter has a torsional nature. In this interpretation, J is the generator of $U(1)$ electromagnetic gauge transformations in opposite directions on the left- and right-handed spinor spaces, with similar transformations, but in the same sense, generated by γ^5 as the spinor analogs of duality transformations.

In Section 3, the phase transformations are related to the properties of almost quaternionic and quaternionic structures, and the physical and geometric significance of the $U(3)$ group of duality transformations is discussed, with a brief reference to a relationship with instantons and global topology.

The geometric approach to quantization is completed by Kostant's method of quantizing real valued scalar functions on a symplectic manifold. In the present case this employs the line bundle with structure group $GL(2, \mathbb{C})/SL(2, \mathbb{C}) \approx \mathbb{C}^*$ and curvature form given by the contraction of the Ricci curvature with the complex structure tensor. For the space-time application, this approach is equivalent to that of Segal for the special case of Einstein spaces; but more generally a distorted coordinate basis is involved. Section 4 of this paper outlines the Kostant technique, and shows how duality rotations transform particle states into antiparticle states, and how a stringlike spectrum appears in geometric quantization.

2. DUALITY ROTATIONS

The central feature in the relation of $T_{\mu\nu}$ to internal symmetry is the classification of $T_{\mu\nu}$ itself by its eigenvectors (Plebanski, 1964; Petrov, 1969). Adopting Plebanski's notation there are four types of $T_{\mu\nu}$ denoted by $[\lambda_0\text{-}\lambda_1\text{-}\lambda_2\text{-}\lambda_3]_{(n)}$, where λ_a for $0 \leqslant a \leqslant 3$ are the eigenvalues of $T_{\mu\nu}$ and n is the degree of the minimal polynomial. If complex eigenvalues are denoted by Z, real eigenvalues whose eigenspaces contain timelike eigenvectors by T, real eigenvalues with null but not timelike eigenvectors by N, and real eigenvalues with only spatial eigenvectors by S, then the four types of $T_{\mu\nu}$ are $[T\text{-}S_1\text{-}S_2\text{-}S_3]_{(4)}$, $[Z\text{-}\bar{Z}\text{-}S_1\text{-}S_2]_{(4)}$, $[2N\text{-}S_1\text{-}S_2]_{(4)}$, and $[3N\text{-}S]_{(4)}$ each of which gives rise to degenerate subtypes when two or more of the distinct eigenvalues are equal. This scheme is not quite unique, but has been related to such a scheme in terms of Lorentz orbits by Collinson and Shaw (Collinson and Shaw, 1972). If the validity of Einstein's field equation

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = kT_{\mu\nu} \tag{2.1}$$

is assumed, then the geometric significance of the eigenvalues is that they determine the principal curvatures of the Einstein tensor, whilst their sum determines the scalar curvature.

The flat-space, free Poincaré covariant quantum mechanical particle approach to the coupling of matter to geometry by preferred frames (Lloyd-Evans, 1979) involves the type $[T\text{-}3S]_{(2)}$ for massive particles and the incompletely diagonalizable case $[4N]_{(2)}$ for massless particles. In each case the energy-momentum vector P_μ is an eigenvector, and in the massive case, therefore, $T_{\mu\nu}$ is diagonalized by being put in its rest frame; the three spatial eigenvalues are each zero, whilst the timelike eigenvalue T is given by the mass-energy density ρ, and the $SO(3, \mathbb{R})$ little group freedom is replaced by a corresponding freedom of choice of spatial eigenvectors.

Because these are degenerate members of the families $[T\text{-}S_1\text{-}S_2\text{-}S_2]_{(4)}$ and $[2N\text{-}S_1\text{-}S_2]_{(4)}$, attention will be concentrated on these two in quoting Plebanski's results on how $T_{\mu\nu}$ determines the vierbein. In the former case the four eigenvectors V^a, $0 \leqslant a \leqslant 3$, constitute an orthonormal tetrad and hence a 16-component vierbein $L_\mu{}^a$ such that

$$L_\mu{}^a = V_\mu{}^a \tag{2.2}$$

$$g_{\mu\nu} = L_\mu{}^a L_\nu{}^b \eta_{ab} \tag{2.3}$$

$$T_{\mu\nu} = T L_\mu{}^0 L_\nu{}^0 - S_1 L_\mu{}^1 L_\nu{}^1 - S_2 L_\mu{}^2 L_\nu{}^2 - S_3 L_\mu{}^3 L_\nu{}^3 \tag{2.4}$$

where $V_\mu{}^a$ is the μth component of the eigenvector V^a, $g_{\mu\nu}$ is the metric tensor, and η_{ab} the Minkowski metric. For $[2N\text{-}S_1\text{-}S_2]_{(4)}$ the two spatial eigenvectors determine vierbein components in a similar way, but the double eigenvalue N has only one eigenvector t_μ, which must be a null vector and can be expressed as $t_\mu = \frac{1}{2}(L_\mu{}^0 + L_\mu{}^3)$. The remaining vector spanning the vectorial subspace of the eigenvector N is any linear combination of t_μ and of $s_\mu \equiv \frac{1}{2}(L_\mu{}^0 - L_\mu{}^3)$ and can be chosen as ϵs_μ with $\epsilon = \pm 1$, when $T_{\mu\nu}$ becomes

$$T_{\mu\nu} = 2\epsilon t_\mu t_\nu + N L_\mu{}^0 L_\nu{}^0 - N L_\mu{}^3 L_\nu{}^3 - S_1 L_\mu{}^1 L_\nu{}^1 - S_2 L_\mu{}^2 L_\nu{}^2 \tag{2.5}$$

The complex structure tensor J is determined by eight of the ten components of the pseudosymmetric vierbein $[L_\mu{}^a\}$, which is related to the 16-component vierbein $L_{\mu\nu}$ by

$$L_\mu{}^a = \{L_\mu{}^b\} O_b{}^a \tag{2.6}$$

where $O_b{}^a$ is a Lorentz vierbein rotation defined in the space of eigenvectors. The condition of pseudosymmetry (Isham et al., 1970), i.e.,

$$\{L_\mu{}^a\}\eta_{a\nu} = \{L_\nu{}^a\}\eta_{a\mu}$$

imposes a unique relationship between coordinates and frames that eliminates the freedom of vierbein rotations. Physically, J is the generator of phase transformations of spinors and of simultaneous rotations in the $l - n$ and $m - \bar{m}$ planes of the null tetrad $(l_\mu, m_\mu, \bar{m}_\mu, n_\mu)$ defined through

$$L_\mu{}^0 = l_\mu + n_\mu, \qquad L_\mu{}^3 = l_\mu - n_\mu$$
$$L_\mu{}^1 = m_\mu + \bar{m}_\mu, \qquad L_\mu{}^2 = i(m_\mu - \bar{m}_\mu) \tag{2.7}$$

and one of the further aims of this section is to consider its relation to electromagnetic gauge transformations. Before doing so, however, we examine the degenerate case $[T\text{-}3S]_{(2)}$ for which spatial eigenvalues are equal; in this case Plebanski showed that the vierbein $L_\mu{}^a$ is no longer unique and instead is defined only up to an arbitrary rotation $O_b{}^a \in SO(3, \mathbb{R})$ in the space of spatial eigenvectors. Inverting (2.7) to obtain the null tetrad shows that there is a similar $SO(3, \mathbb{R})$ indeterminacy in the choice of J and this situation for a $4n$-dimensional manifold is characterized geometrically as the existence of an almost quaternionic structure, and the consequences of this will be considered in the next section.

In order to investigate the possible electromagnetic phase interpretation of the J phase, $T_{\mu\nu}$ is split into its trace and trace-free part $U_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{4}T_\alpha{}^\alpha g_{\mu\nu}$, which determines the tensor analog of the Penrose curvature spinor $\Phi_{AB\dot{C}\dot{D}}$, $1 \leqslant A, B, C, D \leqslant 2$. Plebanski showed that this could be expanded in terms of a basis f^k_{AB}, $1 \leqslant k \leqslant 3$ for the space of symmetric spinor of type $(1, 0)$ as

$$\Phi_{AB\dot{C}\dot{D}} = \sum_{k=1}^{3} \epsilon_n f^k_{AB} f^k_{\dot{C}\dot{D}} \tag{2.8}$$

where the basis can be chosen so that $\epsilon_k = 0, \pm 1$ and the number of ϵ_k which are zero has an invariant significance as the dimension of the subspace annihilated under the antilinear transformation on the basis by $\Phi_{AB\dot{C}\dot{D}}$

$$\Phi^{AB}_{\dot{C}\dot{D}} f^{k\dot{C}\dot{D}} = \epsilon_k f^k_{\dot{C}\dot{D}} f^{k\dot{C}\dot{D}} f^{kAB} \tag{2.9}$$

Using the spin tensor $S^{AB}_{\mu\nu}$ which maps antisymmetric tensors on $\mathbb{R}^4$ onto the space $\mathbb{C}^3$ of symmetric spinors and their complex conjugates, one may define a quantity $f^k_{\mu\nu}$ by

$$f^k_{\mu\nu} = S^{AB}_{\mu\nu} f^k_{AB} + S^{\dot{C}\dot{D}}_{\mu\nu} f^k_{\dot{C}\dot{D}} \tag{2.10}$$

from which may be defined three formal electromagnetic energy-momentum tensors $E^k_{\mu\nu}$:

$$E^k_{\mu\nu} = f^k_{\mu\lambda} f^{k\lambda}_{\nu} - \tfrac{1}{4} g_{\mu\nu} f^k_{\alpha\beta} f^{k\alpha\beta} \tag{2.11}$$

Equations (2.8)–(2.11) enable the trace-free energy-momentum tensor to be expressed as the sum of three formal electromagnetic energy-momentum tensors:

$$U_{\mu\nu} = \sum_{k=1}^{3} \epsilon_n E^k_{\mu\nu} \tag{2.12}$$

The situation in which $\epsilon_k = +\delta_\nu^a$ for one specific a, $1 \leqslant a \leqslant 3$, is the case of Rainich theory, and there are two distinct $T_{\mu\nu}$ types for which this occurs, namely $[4N]_{(2)}$ and $[2T\text{-}2S]_{(2)}$. The distinction between these two is Lorentz invariant, with the former giving null electromagnetism characterized by a Ricci tensor $R_{\mu\nu}$ that is the product of null vectors k_μ,

$$R_{\mu\nu} = k_\mu k_\nu \tag{2.13}$$

and having only three linearly independent components as against the five of the general (latter) case, while the special case of self-dual electromagnetic fields has $T_{\mu\nu} = 0$. Of these three types, $[4N]_{(2)}$ admits a direct interpretation as an electromagnetic wave propagating at the velocity of light (Misner and Wheeler, 1957) and it is this type which occurs in the Poincaré covariant geometric coupling scheme, but the interpretation of $T_{\mu\nu}$ of type $[4N]_{(2)}$ as due to electromagnetism is therefore not unique, in particular, 2-component neutrinos can contribute (Inomata and McKinley, 1971), while parity-conserving 4-component neutrinos also give type $[2T\text{-}2S]_{(2)}$ (Inomata and McKinley, 1965), and one might speculate that this distinction is relevant to the parity problems of the Weinberg–Salam theory of weak and electromagnetic interactions as applied to atomic transitions. For Rainich theory the electromagnetic field (respectively, neutrino field) is only defined up to a duality transformation $e^{i\alpha}$ (respectively, γ^5 transformation) by equation (2.11) and for $[T\text{-}3S]_{(2)}$ only up to a generalized duality rotation belonging to $U(3)$ (Plebanski, 1964), and to interpret these results it is necessary to consider the relation between duality and phase transformations.

For electromagnetism there are a priori four kinds of $U(1)$ transformations to be considered in this context:

1. The duality rotations of electromagnetism

$$F_{\mu\nu} \rightarrow {}^*F_{\mu\nu} \sin\alpha + F_{\mu\nu} \cos\alpha \tag{2.14}$$

where

$${}^*F_{\mu\nu} = \tfrac{1}{2}\epsilon_{\kappa\lambda\mu\nu} F^{\kappa\lambda} \tag{2.15}$$

or in spinor form

$$\varphi_{AB} \rightarrow e^{i\alpha}\varphi_{AB} \tag{2.16}$$

where

$$F_{\mu\nu} = S_{\mu\nu}^{AB}\varphi_{AB} + S^{\dot{C}\dot{D}}\tilde{\varphi}_{\dot{C}\dot{D}} \tag{2.17}$$

2. The phase rotations generated by the complex structure J used in the definition of spinors consistent with the null tetrad structure of equation

(2.7), for which $J_\nu{}^\mu$ has the real representation

$$\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

3. In quantum mechanics another set of phase transformations are generated by the complex structure tensor J^1 of the 1-particle phase space with symplectic form $\Omega^1 = dp \wedge dq$, where J^1 is related to Ω^1 through the fact that infinitesimal symplectic transformations, A, satisfy

$$A^\dagger J^1 + J^1 A = 0$$

where $A^\dagger$ is the transpose of A, whereas complex transformations satisfy

$$AJ^1 + J^1 A = 0$$

This complex structure tensor J^1 is the real representation on $\mathbb{R}^8$ phase space of the pure imaginary number (i) appearing in the representation exp (ipq) of the transformation group $\mathbb{R}^4$.

4. The $U(1)$ phase invariance of electromagnetic gauge theories. Of these, (4) was originally introduced by Weyl in relation to the phase invariance of quantum mechanical matrix elements, which when applied to the Dirac equation entailed the replacement of the ordinary derivative ∂_μ by the covariant derivatives $\partial_\mu - ieA_\mu$, where the gauge field is interpreted as electromagnetism. In the original argument the phase transformation was applied to a Dirac spinor, and subsequently the geometric significance of such transformations together with the physical meaning of (i) have been investigated by Hestenes (1967, 1975). That investigation showed that the factor (i) used in defining complex 4-component spinors could be represented in the real Dirac algebra by γ^5, while the similar factor, denoted (i'), appearing in the definition of 2-component spinors is the pseudoscalar of the Pauli algebra. Furthermore, he showed that with these definitions the factor (i) in the plane wave states exp (ipx) is $i\sigma_3$, where $\sigma_3 \equiv \gamma_0\gamma_3$, as also is the factor (i) in the electromagnetic covariant derivative, so that in the real Dirac algebra both the electromagnetic phase transformations and the rotations of the 1-particle phase space in (3) above are generated by $\gamma_1 \wedge \gamma_2$ or, equivalently, by $i\gamma_0 \wedge \gamma_3$. Geometrically such phase transformations are represented by rotations in the null planes of the light cone in terms of which the spinors are defined, and J generates simultaneous rotations through the same angle in both planes.

An a priori alternative interpretation is to take $U(1)$ as the group of automorphisms of complex numbers consisting of phase rotations. If this group is used to extend the group D_2 of rotations through π about the x, y, and z axes, then the Pauli 4-group of spin matrices together with phase rotation is the nontrivial extension, and the basis vector of the spin representation is geometrically described as a Cartan isotropic vector (x^i) whose two independent components (ζ^1, ζ^2) are given by

$$\zeta^1 = [(x^1 + ix^2)/2]^{1/2}, \qquad \zeta^2 = [(x^1 - ix^2)/2]^{1/2}$$

where

$$\sum_{i=1}^{3} (x^i)^2 = 0$$

The phase angle θ of the spinors is then given by

$$\theta = \arctan (x^2/x^1)$$

If Lorentz boosts of velocity from 0 to c are included, a second spinor basis is obtained with phase corresponding to a hyperbolic rotation, conventionally in the 0-3 plane, and further addition of parity gives back the γ-matrix formalism.

These results strongly suggest that the electromagnetic $U(1)$ gauge invariance group should be described geometrically in terms of these spinor phase rotations, and so we consider the possibility of deriving the Ricci tensor of electromagnetism directly from the connection associated with J-phase invariance. J itself is a tensor field on space-time with antisymmetric coefficients $(J_\lambda{}^\kappa)$ in an orthonormal basis, and the condition the $U(1)$ gauge field A_μ can be expressed in the usual form iA_μ is that A_μ be an eigenfunction of J, i.e., $J_\lambda{}^\kappa A_\kappa = \pm iA_\lambda$, which means that A_μ is to be geometrically quantizable and so either holomorphic or antiholomorphic for Euclidean space or a future or pasting pointing null vector in Minkowski space, where the complexified spaces are two dimensional with $(a + ib)X \equiv (a + bJ)X$; $a, b \in \mathbb{R}$, $X \in \mathbb{R}^4$.

The most general linear geometric connection $\Gamma^\kappa_{\lambda\mu}$ has the form

$$\Gamma^\kappa_{\lambda\mu} = \{{}^{\kappa}_{\lambda\mu}\} + T^\kappa_{\lambda\mu} + Q^\kappa_{\lambda\mu} \tag{2.18}$$

where $\{{}^{\kappa}_{\lambda\mu}\}$ are the pseudo-Riemannian connection coefficients, $T^\kappa_{\lambda\mu}$ is a torsional term, and $Q^\kappa_{\lambda\mu}$ is defined by

$$Q^\kappa_{\lambda\mu} = \nabla'^\kappa g_{\lambda\mu}$$

with ∇' denoting the total covariant derivative in (2.18). Because the metric is Hermitian with respect to J, it has zero J charge, and does not couple to the connection determined by the gauge field A_μ, which can therefore only belong to the torsional term in (2.18). For spinors, the coupling to A_μ will in general be nonzero and this is the connection noted previously for such objects by Fock (1929). The total Riemannian and Ricci curvatures, $R^\kappa_{\nu\mu\lambda}$ and $R_{\mu\lambda}$ are derived from their analogs $K^\kappa_{\nu\mu\lambda}$ and $K_{\mu\lambda}$ of the pseudo-Riemannian connection by

$$R^\kappa_{\nu\mu\lambda} = K^\kappa_{\nu\mu\lambda} + 2\nabla_{[\nu}T_{\mu]\lambda}{}^\kappa + 2T_{[\nu|\rho|}{}^\kappa T_{\mu]\lambda}{}^\rho \tag{2.19}$$

$$R_{\mu\lambda} = K_{\mu\lambda} + \nabla_\kappa T^\kappa_{\pi\nu} - \nabla_\mu T^\kappa_{\kappa\lambda} + T^\kappa_{\kappa\rho}T^\rho_{\mu\lambda} - T^\kappa_{\mu\rho}T^\rho_{\kappa\lambda} \tag{2.20}$$

where the square brackets denote antisymmetrization, the notation $[\nu|\rho|\cdots\mu]$ indicates the exclusion of ρ from this antisymmetrization, and ∇ denotes covariant derivation with respect to the $\{^\kappa_{\lambda\mu}\}$. For J-phase transformations the torsion takes the form

$$T^\kappa_{\lambda\mu} = J_{[\lambda\mu]}A^\kappa - J_{[\mu}{}^\kappa A_{\lambda]} + J^\kappa{}_{[\lambda}A_{\mu]} \tag{2.21}$$

where $J_{[\lambda\mu]}$ correspond to the coefficients of the Kähler form $g(X, JY)$, defined by $J_{[\lambda\mu]} = J_\lambda{}^\kappa g_{\kappa\mu}$. In equation (2.19) it is the second term which takes the place of the electromagnetic field $F_{\nu\mu}$, and if only the electromagnetic connection occurs, this reduces to the usual expression, but more generally the metric terms in (2.21) lead to nonvanishing of the quadratic torsion term. The electromagnetic $U(1)$ gauge field, however, is not the only source of torsion, the other being the $SO(1, 3)$ vierbein gauge field, and in general both have to be treated together, subject to the mutual consistency condition for a Hermitian space that the total covariant derivative of the complex structure tensor field $J_\lambda{}^\kappa$ vanishes. The limitations this imposes on the torsion have been considered by Yano (1965); here we note only that one of the special cases permitted is that of a complex semisymmetric connection, and that in this case $R_{\mu\lambda}$ contains a contribution of the general form $(S_\mu S_\lambda - S_\rho S^\rho g_{\mu\lambda})$ from the quadratic torsion term, where

$$S_\mu \equiv \tfrac{1}{3}[\Gamma^\kappa_{\kappa\mu} - \Gamma^\kappa_{\mu\kappa}] \tag{2.22}$$

If the torsion is covariant constant and S^μ a null vector, then this term gives directly the null Rainich form of electromagnetism (2.13). On the other hand, it is clear that the general electromagnetic energy-momentum tensor of type $[2T\text{-}2S]_{(2)}$ does not arise directly from the gauge field but can only occur through the Ricci tensor for the symmetric part of the connection; furthermore in Inomata and McKinley's 2- and 4-component neutrino interpretation of $[4N]_{(2)}$ and $[2T\text{-}2S]_{(2)}$ types of $T_{\mu\nu}$ it is shown that only the former can be related to a Heisenberg nonlinear spinor equation and hence to a torsional interpretation (Rodichev, 1961).

In Rainich theory the electromagnetic field is only defined up to an arbitrary rotation, and conventionally a duality complexion α is chosen so that the nonnull field is purely electric in a local Minkowski reference frame, in which case Maxwell's equations require that

$$\alpha_\mu = \alpha_{,\mu} \tag{2.23}$$

where

$$\alpha_\mu \equiv (-g)^{1/2}\epsilon_{\mu\kappa\lambda\nu}R^{\kappa\delta;\lambda}R_\delta{}^\nu/R_{\sigma\tau}R^{\sigma\tau} \tag{2.24}$$

in the general $[2T\text{-}2S]_{(2)}$ case, while for $[4N]_{(2)}$ α_μ is not defined, and duality transformations rotate the polarization vector of the field about the direction of propagation, but in the neutrino interpretation the derivative $\beta_{,\mu}$ of the γ^5 complexion β behaves like a torsion vector in the Heisenberg interpretation.

The relationship between phase and duality rotations for $[4N]_{(2)}$ electromagnetism is shown by Ludwig's definition of the curvature spinor $\Phi_{AB\dot{C}\dot{D}}$ in terms of 2-component spinors (Ludwig, 1970):

$$\Phi_{AB\dot{C}\dot{D}} = b\kappa_A\kappa_B\kappa_{\dot{C}}\kappa_{\dot{D}} \tag{2.25}$$

where b is a constant. A spinor of type (1, 0), φ_{AB}, is defined from the κ_A in terms of a duality complexion α:

$$\varphi_{AB} = \kappa_A\kappa_B e^{-i\alpha} \tag{2.26}$$

Under a J-phase transformation through θ the dotted and undotted spinors κ and $\bar{\kappa}$ are multiplied by $\exp(\mp i\theta/2)$, while the duality phase undergoes a transformation $\alpha \to \alpha + \theta$, with the result that φ_{AB} is phase invariant and has no electric charge if the J-phase generates the electromagnetic $U(1)$ gauge group.

3. INTERNAL SYMMETRY

In this section we consider the more highly symmetric types of $T_{\mu\nu}$ and the physical interpretation of the symmetries. The most symmetric of all is the vacuum $T_{\mu\nu}$, of the type $[4T]_{(1)}$ in Plebanski's notation, and the type of greatest interest here is the type $[T\text{-}3S]_{(2)}$. In this case the vierbein analysis (2.2) and its relation to the complex structure (2.7) shows that there are three linearly independent complex structures differing by $SO(3, \mathbb{R})$ vierbein rotations; such a situation is characterized geometrically as an almost quaternionic structure (Bonan, 1967; Kraines, 1966; Yano and Ako, 1973; Ishihara, 1974), and in the following we make use of a number of properties proven in these references.

An almost quaternionic structure in four dimensions can also be defined as a reduction of the $GL(4, \mathbb{R})$ group of general linear frame transformations to $GL(1, \mathbb{H})$ or as a manifold with local holonomy group $Sp(1) \times Sp(1)$. The quaternionic structure is integrable if in every coordinate neighborhood the three linearly independent complex structures J_1, J_2, and J_3 can be expressed in the canonical form (3.1) below:

$$J_1 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad J_2 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$J_3 = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{3.1}$$

If the almost quaternionic structure is not integrable, but admits one integrable section, the complex structure associated with the light cones, then for this case nonintegrability is equivalent to the existence of a nontrivial $SO(3, \mathbb{R})$ vierbein gauge field which is interpreted geometrically as torsion. Integrability has the very important physical limitation that the Ricci tensor, and hence also $T_{\mu\nu}$, must vanish identically, so for the present classification it is the nonintegrable case that is of interest. The integrable case has, however, been of great interest recently in the study of instanton solutions of the self-dual Yang-Mills field equations (Belavin et al., 1975), for which $T_{\mu\nu}$ is necessarily zero, and their topological classification; and a brief comparison of this with the eigenvalue classification of the conformal curvature is given later in this section. The one remaining point of major physical significance is that unlike the case of a single complex structure which commutes with $SL(2, \mathbb{C})$ Lorentz spinor transformations, the almost quaternionic structure cannot be defined in a Lorentz frame covariant manner and commutes only with $SO(3, \mathbb{R})$ rotations so that any physical symmetry defined in terms of quaternionic phase transformation must be broken.

For $T_{\mu\nu}$ type $[T\text{-}3S]_{(2)}$ the trace-free part $U_{\mu\nu}$ was shown by Plebanski to be invariant under $U(3)$ generalized duality rotations, and we consider these in relation to the $Sp(1) \times Sp(1)$ group of quaternionic phase transformations and vierbein rotations. Using an orthonormal basis for the space of symmetric spinors of types (1, 0) and (0, 1) the Penrose spinor $\Phi_{AB\dot{C}\dot{D}}$ can be expressed in terms of the trace-free eigenvalues as

$$\Phi_{AB\dot{C}\dot{D}} = \tfrac{1}{2}(T - S_3)(\overset{1}{q}_{AB}\overset{1}{q}_{\dot{C}\dot{D}} + \overset{2}{q}_{AB}\overset{2}{q}_{\dot{C}\dot{D}}) + \tfrac{1}{2}(S_2 - S_1)(\overset{1}{q}_{AB}\overset{2}{q}_{\dot{C}\dot{D}} + \overset{2}{q}_{AB}\overset{1}{q}_{\dot{C}\dot{D}})$$
$$+ (T + S_3)(\overset{3}{q}_{AB}\overset{3}{q}_{\dot{C}\dot{D}}) \tag{3.2}$$

where the eigenvalues are subject to the trace-free condition

$$T = \sum_{i=1}^{3} S_i = 0$$

$$q^1_{AB} = -\frac{\iota}{2^{1/2}}(\kappa_A\kappa_B - \iota_A\iota_B) \qquad q^2_{AB} = \frac{1}{2^{1/2}}(\kappa_A\kappa_B + \iota_A\iota_B) \tag{3.3}$$

$$q^3_{AB} = \frac{\iota}{2^{1/2}}(\eta_A\iota_B - \iota_A\kappa_B)$$

where (κ_A, ι_B) is a 2-component spinor basis associated with the null tetrad (2.7). For $[T\text{-}3S]_{(2)}$ each $S_i = S$ and only the diagonal terms remain. By virtue of the manifest Hermiticity of $\Phi_{AB\dot{C}\dot{D}}$ it is invariant under the J-phase rotations where J is the complex structure used to define the spinors. The space of symmetric spinors is isomorphic to the $\mathbb{R}^6$ space of bivectors $V^a \wedge V^b$, $0 \leqslant a, b \leqslant 3$ on which the absolute involution tensor $\epsilon_l{}^k$, $1 \leqslant k, l \leqslant 6$ defined by

$$\epsilon_k{}^l \epsilon_l{}^m = -\delta_k{}^m \tag{3.4}$$

$$(\epsilon^2 + 1)^3 = 0 \tag{3.5}$$

determines a complex structure which generates the duality transformations. A nondegenerate Hermitian form of maximal rank is invariant under $GL(3, \mathbb{C}) \cap O(3, 3) \equiv U(3)$ transformations of $\mathbb{R}^6$ and this is Plebanski's group of generalized duality rotations. The $SO(3, \mathbb{R})$ vierbein group acts on both $\{V^0 \wedge V^i\}$ and $\{V^i \wedge V^j\}$, $1 \leqslant i, j \leqslant 3$, so that the $SO(3, \mathbb{C})$ subgroup of $U(3)$ consists of left- and right-handed $SO(3, R)$ transformations, where $V^a \wedge V^b$ are the eigenvectors of the tensor analog of the Weyl-like spinor

$$V_{ABCD} \equiv \Phi_{\dot{R}\dot{S}(AB}\Phi^{\dot{R}\dot{S}}{}_{CD)} \tag{3.6}$$

and the $U(3)$ group of the Weyl spinor was suggested by Sarfatti (1975) as the origin of internal symmetry. The diagonal elements are the changes of complexion in the three orthogonal electrodynamics defined by (2.11), the product of all three being generated by $\epsilon_l{}^k$, which is the unit operation in $U(3)$. Electric charge is assumed to be an eigenvalue of the operator J, and as such is not a member of this $U(3)$ group, although it does generate a change of complexion according to the definition in (2.26). Denoting a bivector basis of $\mathbb{R}^6$ by $\{E, H]$ the action of each complex tensor J_a of (3.1) is a map:

$$\{E_a, H_a\} \mapsto \{E_a, H_a\}$$
$$\{E_b, H_b\} \mapsto \{-H_b, -E_b\} \tag{3.7}$$
$$1 \leqslant a \neq b \leqslant 3$$

The tensor J_a evidently does not act as a complex structure on $\mathbb{R}^6$; instead this role is played by the involution tensor $\epsilon_l^{\,k}$, that corresponds to the Hodge duality operator on $\mathbb{R}^4$. Each complex structure J_a can be used to define a 4-form $dz^1 \wedge dz^2 \wedge d\bar{z}^1 \wedge d\bar{z}^2$, and hence a duality operator, but since in four dimensions there is only one class of 4-forms, there is essentially only one such operator. The three generalized duality operators which are diagonal in $U(3)$ can be expressed in local coordinates by

$$\epsilon^{\alpha\beta}_{\gamma\delta} \equiv (x^\alpha \wedge x^\beta) \frac{\partial}{\partial(x^\alpha \wedge x^\delta)} - (x^\gamma \wedge x^\delta) \frac{\partial}{\partial(x^\alpha \wedge x^\beta)}$$

$$0 \leqslant \alpha \neq \beta \neq \gamma \neq \delta \leqslant 3$$

The three linearly independent pairs of bivectors, $(x^\alpha \wedge x^\beta)$ and $(x^\gamma \wedge x^\delta)$, can be chosen so as to define the planes of phase and hyperbolic phase rotations generated by each of the three complex structures. The other generators of $U(3)$ are the $SO(3, \mathbb{C})$ transformations of $\mathbb{R}^6$ induced by the vierbein rotations, so the group does not contain any elements induced by pure Lorentz boosts, and its Lorentz-covariant generalization is just the $O(3, 3)$ symmetry group for $U_{\mu\nu} = 0$, with Euclidean analog $O(6)$. In relation to the unitary groups of hadron physics, it may be noted that the spinor group of the latter is $SU(4)$, whose fundamental representation is spanned by null hyperplanes of $\mathbb{R}^6$.

The full symmetry of the vacuum includes Lorentz transformations on spaces of opposite 4-orientation, so we consider how this is affected by the curvature: the 4-orientation of space-time can be defined by a nonvanishing 4-form, and in terms of the curvature, the Euler density $(32\pi^2)^{-1} \sum_{\kappa,\lambda,\mu,\nu} \epsilon^{\kappa\lambda}_{\mu\nu} \Omega_\kappa{}^\mu \wedge \Omega_\lambda{}^\nu$ and the Pontryagin density $-(8\pi^2)^{-1} \operatorname{Tr} \Omega^2$ define 4-forms, although not necessarily nonvanishing, where

$$\Omega_\lambda{}^\kappa = \frac{1}{2} \sum_{\mu,\nu=0}^{3} R^\kappa_{\lambda\mu\nu}\, dx^\mu \wedge dx^\nu$$

These two quantities, whose integrals are topological invariants, have been recently used to define and classify instanton solutions to the self-dual Yang–Mills field equations (Belavin et al., 1975), which according to the analysis of the previous section arise from an essentially torsional connection associated with quaternionic phase invariance. Here we point out that Euler and Pontryagin densities from the symmetric part of the connection can readily be calculated from the eigenvectors of the curvature of an Einstein space (Petrov, 1969). Using the expressions

$$\begin{bmatrix} M & N \\ N & M \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} M & N \\ N & -M \end{bmatrix}$$

for the bivector curvature R_{ab}, $1 \leqslant a, b \leqslant 6$, of Euclideanized and Petrov type 1 pseudo-Riemannian spaces with

$$M = \begin{bmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} \beta_1 & 0 & 0 \\ 0 & \beta_2 & 0 \\ 0 & 0 & \beta_3 \end{bmatrix}$$

where

$$\sum_{i=1}^{3} \alpha_i = -\tfrac{1}{4}R, \qquad \sum_{i=1}^{3} \beta_i = 0$$

and R is the scalar curvature, the Euler and Pontryagin densities for the two cases are, respectively,

$$(4\pi^2)^{-1} \sum_{i=1}^{3} (\alpha_i^2 + \beta_i^2) \qquad \text{and} \qquad \pi^{-2} \sum_{i=1}^{3} \alpha_i\beta_i$$

and

$$(4\pi^2)^{-1} \sum_{i=1}^{3} (\alpha_i^2 + \beta_i^2) \qquad \text{and} \qquad 0$$

Imposing the condition that the curvature bivectors be $SO(3, R)$ invariant entails the requirement

$$\alpha_i = -(12)^{-1}R, \qquad \beta_i = 0 \qquad 1 \leqslant i \leqslant 3$$

and also eliminates consideration of Petrov types II and III for pseudo-Riemannian space. Self-duality of the Yang–Mills field equations requires integrability of the quaternionic structure and hence vanishing of the conformal curvature of the symmetric connection as well as of the Ricci curvature. For the almost quaternionic case instanton solutions do not exist, and in general the $U_{\mu\nu}$-free curvature is not expressible in the canonical form above in terms of the bivectors constructed from the eigenvectors of $U_{\mu\nu}$, so that no neat topological classification of massive Yang–Mills fields is possible in general, but there is one exception to this, namely where the Weyl spinor W_{ABCD} is algebraically dependent on $U_{\mu\nu}$ through a relation like (3.6):

$$W_{ABCD} = w\Phi_{\dot{R}\dot{S}(AB}\Phi^{\dot{C}\dot{D}}_{RS)} \tag{3.8}$$

where w is a scalar. In this special case the Euler and Pontryagin densities can be obtained from the eigenvalues of the almost quaternionic Ricci tensor. Unlike the conformal curvature, R_{ab}, $1 \leqslant a, b \leqslant 6$, can be fully diagonalized in terms of the basis $\{V^0 \wedge V^i, V^i \wedge V^j\}$ of bivectors constructed from the eigenvectors V^μ $(0 \leqslant \mu \leqslant 3)$ of the Ricci curvature tensor, and for the almost quaternionic case this gives a curvature matrix

$$\begin{bmatrix} M & 0 \\ 0 & L \end{bmatrix}$$

with $M = (\frac{1}{3}T - \frac{1}{12}R)I_3$ and $L = -(\frac{1}{3}T + \frac{1}{12}R)I_3$, where I_3 is the unit matrix, for the Euclidean case and

$$\begin{bmatrix} M & 0 \\ 0 & -L \end{bmatrix}$$

for the Minkowski space, so the Pontryagin density of the symmetric part of the connection is zero in each case.

4. KOSTANT QUANTIZATION

This section considers how J-phase and quaternionic phase transformations are related to geometric quantization. Firstly in the finite-dimensional analog of Segal's method using the pseudo-Riemannian metric and the complex structure J, J-phase transformations preserve commutation and anticommutation relations so that J charge is simultaneously specifiable with the spin quanta of the Segal-type scheme. If the metric is Hermitian with respect to a quaternionic structure, this argument extends to the quaternionic transformations $Sp(1) \times Sp(1)$. This type of quantization is of a static character and to obtain a dynamical quantization it is necessary to use the Kostant method based on the Ricci curvature instead.

The Kostant theory here quantizes fluctuations of real scalar fields in terms of the Ricci curvature form of the line bundle with structure group $GL(2, \mathbb{C})/SL(2, \mathbb{C})$ chosen so that its Ricci form coincides with that derived from the Ricci curvature of space-time including the torsional contributions compatible with integrability of J. This quantization is characterized by the cohomology class of the Ricci 2-form, so that the Segal-type quantization is only equivalent to the Kostant type if the Ricci and Kähler forms $R(X, JY)$ and $g(X, JY)$ are cohomologically equivalent, i.e., if $U_{\mu\nu}$ is a pure divergence, and nontrivial dynamics are associated with the breakdown of this condition.

The Kostant theory requires the existence of locally Hamiltonian fields that are defined as vector fields X on the space-time M such that their Lie derivatives of the Ricci form $R(V, JY)$ vanish:

$$\mathscr{L}_X R(V, JY) = 0 \tag{4.1}$$

This requires $\mathscr{L}_X V = 0 = \mathscr{L}_X Y$ if symmetries of the Ricci form are to coincide with those of the Ricci tensor. Since the Lie derivative $\mathscr{L}_X$ can be expressed as (4.2)

$$\mathscr{L}_X = d \cdot i_X + i_X \cdot d \tag{4.2}$$

where d denotes exterior derivation and i_X contraction with respect to X, it follows, using the Bianchi identity, that (4.1) is satisfied if $i_X[R(V, J(Y))]$ is a closed 1-form $d\varphi$, and it is the map $\varphi \mapsto X_\varphi$ satisfying the latter condition which is the basis of the Kostant quantization.

A real valued scalar function φ on M is said to be prequantized by the map

$$\varphi \mapsto \delta_\varphi \tag{4.3}$$

where δ_φ acts by covariant derivation on sections $s \in \Gamma(L)$, where $\Gamma(L)$ denotes the space of sections of the line bundle L on M with connection ∇, by

$$\delta_\varphi s = (\nabla_{X_\varphi} - 2\pi i\varphi)s \tag{4.4}$$

where X_φ is a locally Hamiltonian field related as above to φ, and the second term in (4.4) is included to eliminate the curvature and so ensure that (4.3) can lead to a Lie algebra homomorphism, but also leads to the representation being reducible. To avoid this, a polarization $F_x(M)$ of the tangent space $T_x(M)$ is chosen so that the sections $s \in \Gamma(L)$ are covariant constant with respect to directions in $F_x(M)$ at each $x \in M$.

For the present application there is a natural choice of polarization, the Kähler polarization, based on the subspace F_x and $\bar{F}_x$ of holomorphic and antiholomorphic vector fields

$$\begin{gathered} R(X_\varphi, X_\psi) = 0 = R(\bar{X}_\varphi, \bar{X}_\psi) \\ R(X_\varphi, \bar{X}_\psi) \neq 0 \end{gathered} \tag{4.5}$$

where

$$X_\varphi, X_\psi \in F_x(M);\ \bar{X}_\varphi, \bar{X}_\psi \in \bar{F}_x(M);\ \varphi, \psi \in \Gamma(L)$$

To make the space of sections $\Gamma(L)$ into a pre-Hilbert space, it is necessary to impose a square integrability condition, and this depends on introducing a volume element v_x on $F_x(M)$ and considering wavefunctions of the local form

$$\psi = \varphi_x(v_x)^{1/2}, \qquad \varphi \in \Gamma(L) \tag{4.6}$$

and consistent treatment of the sign ambiguity requires use of a double-valued covering $ML(2, \mathbb{C})$ of $GL(2, \mathbb{C})$ subject to a global cohomology condition $H^2(M, Z_2) = 0$, which in this particular case is precisely the requirement for a consistent definition of spinors. The Hilbert space is then the completion of the pre-Hilbert space of square integrable functions W where

$$W = \{\psi \in \Gamma(L \otimes L^F) | \nabla_X \psi = 0\} \forall X \in F_x(M) \tag{4.7}$$

where L^F is the bundle of 1/2-forms associated with the volume element of F. For the present case this is just the space of square integrable holomorphic sections of type (2.0), and using the complex structure J to define a 4-form

$dZ^1 \wedge d\bar{Z}^1 \wedge dZ^2 \wedge d\bar{Z}^2$ and hence a duality operator, it is mapped onto the antiparticle space of antiholomorphic forms of type (0, 2) by this operator. Inclusion of the volume element requires also that the prequantum operator δ_φ in (4.3) be replaced by $\tilde{\delta}_\varphi$, where

$$\tilde{\delta}_\varphi = \delta_\varphi + \mathscr{L}_{X_\varphi} v_x^{1/2} \tag{4.8}$$

In this application the scalar fields φ contribute to the scale of space-time, and the aim is to see what forms of φ are compatible with various symmetries of the Ricci form. Any vector field X on M can be expanded locally in a particular coordinate system without loss of generality as (Kobayashi, 1972)

$$X = \sum_{\mu=0}^{3} \lambda^\mu \, \partial/\partial x_\mu \tag{4.9}$$

where

$$\lambda^\mu = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\mu_1,\ldots,\mu_k=0}^{3} a^\mu_{\mu_1,\ldots,\mu_k} x^{\mu_1} \cdots x^{\mu_k} \tag{4.10}$$

The vector field X generates an automorphism of a kth-order G structure if for each fixed set $(\mu_2, \ldots, \mu_k)$ the element $a^\mu_{\mu_1,\ldots,\mu_k}$ belongs to the Lie algebra $\mathbf{G}$ of G. The Ricci curvature tensor of a symmetric connection can in this way be represented as a third-order structure, but it is more convenient to consider the (ordinary) first-order G-structures in relation to the symmetries mentioned above. In particular, holomorphic vector fields preserve the $gl(2, \mathbb{C})$ structure J so that in (4.10) they have $a^\mu_{\mu_1} \in GL(2, \mathbb{C})$ and using the $GL(4, \mathbb{R})$ expression for J, the J-phase transformations themselves are generated by the vector field

$$\sum_{i=0}^{2} i\left(z^i \frac{\partial}{\partial z^i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i}\right) \tag{4.11}$$

The Ricci form is J invariant and to find what (4.11) can be used to quantize we contract the vector field (4.11) with the simple Ricci form, that for an Einstein space given by $\sum_{i=1}^{2} \frac{1}{2} R dZ^i \wedge d\bar{Z}_i$. This gives

$$d\varphi = \sum_{i=0}^{2} \frac{R}{2} (Z^i \, d\bar{Z}_i + \bar{Z}_i \, dZ^i) \tag{4.12}$$

or

$$\varphi = \sum_{i=0}^{2} \frac{R}{2} (Z^i \bar{Z}_i + \bar{Z}_i Z^i) \tag{4.13}$$

This corresponds formally to a harmonic oscillator whose Hamiltonian φ is quantized by using expression (4.11) in (4.8) for X, and as the Kostant quantization of the harmonic oscillator has previously been investigated (Simms and Woodhouse, 1976) it is not considered here except to point out that the spectrum is formally that of a vibrating string and that here its configuration space is a 2-dimensional subspace of space-time.

For an integrable quaternionic case the Ricci form necessarily vanishes, but the three Kähler forms $g(X, J_a Y)$ can each be used to define a Kostant quantization as above leading to three linearly independent harmonic oscillators rotated into one another by the $SO(3, \mathbb{R})$ vierbein transformations, i.e., a 3-dimensional harmonic oscillator. Such a system has states classifiable in terms of $SU(3)$ representations, but not coinciding with the hadronic $SU(3)$ spectrum.

The simplest non-Einstein space has $U_{\mu\nu}$ of type $[T\text{-}3S]_{(2)}$ and a Ricci form differing from the Einstein case by the inequality of the coefficients $R_{1\bar{1}}$ and $R_{2\bar{2}}$, while the incompletely diagonalizable cases $[2N\text{-}S_1\text{-}S_2]_{(4)}$ and $[3N - S]_{(4)}$ have terms $R_{i\bar{j}}\, dz^i \wedge d\bar{z}^j$ for $i \neq j$ that destroy the simple harmonic oscillator interpretation. This situation will be considered in detail elsewhere in relation to the theory of strings and of minimal surfaces.

5. DISCUSSION

The basic conclusion of this paper is that the geometry of space-time contains far more information of probable relevance to physics than just a classical approximation to the gravitational field, but that physically interesting phenomena are associated with very specialized geometric situations.

The conclusions which can be drawn are of a qualitative nature: Firstly, comparison of the Rainich theory of electromagnetism with the gauge approach showed that the $[2T\text{-}2S]_{(2)}$ and $[4N]_{(2)}$ energy momenta were associated, respectively, with symmetric, and antisymmetric parts of the connection, and also with parity conserving and nonconserving 4- and 2-component neutrinos by the arguments of Inomata and McKinley, and that the torsional gauge group is $U(1) \times SO(1, 3)$. Secondly, extension of these arguments to $[T\text{-}3S]_{(2)}$ leads to an $SO(3, \mathbb{R})$ group of quaternionic phase transformations accompanied by $SO(3, \mathbb{R})$ vierbein transformations with the condition that the $SO(3, \mathbb{R})$ torsional gauge field must be nonzero if the Ricci curvature, and hence also the mass, is not to vanish identically. The maximal symmetry group is that of the vacuum curvature which consists of $O(3, 3)$ transformations on the $\mathbb{R}^6$ space of bivectors, with Plebanski's $U(3)$ as a subgroup preserving the complex structure associated with duality rotations, and characteristic breaking both by the $U_{\mu\nu}$ types and also by the conformal part of the curvature. Since the $O(3\ \ 3)$ group consists in part of chiral transformations it seems clear that this geometric formalism is not capable of describing all known quark flavors, let alone color unless the latter is regarded as a magnetic $SU(3)$. The positive achievement of this work is to show how the phase invariance of the scalar curvature leads to a formal harmonic oscillator or string model, and for quaternionic phase transformations to a three-dimensional harmonic oscillator or a massless 3-string model.

The basic distinction between this and conventional quantum field theory is that this, and arguably any theory of quantum gravity where the curvature of the light cone is built into the definition of the first graviton, is an attempt to produce a nonvacuum (and hence nonmaximal in the sense of rings of W^* operators of the group algebra, or also of not giving a complete commuting set of observables) quantization. In Lagrangian quantum field theory an approach to nonvacuum quantization is made by including Goldstone and Higgs particles in a modified vacuum, and for the present theory J would be represented by Goldstone particles for the breaking of $GL(4, \mathbb{R})$ to $GL(2, \mathbb{C})$, and if the Lagrangian density is identified with the Euler density, then going from $T_{\mu\nu} = 0$ to $T_{\mu\nu} \neq 0$ is represented by adding to the Lagrangian terms which are variously quadratic or quartic in the eigenvectors of $T_{\mu\nu}$ which, therefore, bear a slight resemblance to Higgs particles. The latter analogy is clearly not exact, and zero-mass fields are themselves a source of symmetry breaking for Plebanski's $U(3)$.

This work raises numerous questions, some of which will be investigated elsewhere, notably the use of the Ricci form to describe the dynamics of minimal surfaces or strings, and a modified theory of weak and electromagnetic interactions based on the most general torsional connection. In regard to the latter, the recent renormalization group argument that coupling constants should become equal at short distances weakens the main coupling constant argument against the interpretation of torsion as the weak interactions, although at the expense of requiring some form of geometric quantization.

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